Self-adjointness of Deformed Operators*

Consider the deformed operator A_Θ, with A being an unbounded operator

$$\langle \Psi, A_\Theta \Phi \rangle = (2\pi)^{-n} \lim_{\epsilon \to 0} \iint dx\, dy\, e^{-ixy}\, \chi(\epsilon x, \epsilon y) \langle \Psi, U(y) \alpha_{\Theta x}(A) \Phi \rangle$$

for $\Psi, \Phi \in \mathcal{D}^\infty(A) := \{\Phi \in \mathcal{D}(A) | U(x)\Phi \in \mathcal{D}(A) \text{ is smooth in } \|\cdot\|_{\mathscr{H}}\}$.

Lemma

Assume for A that the following condition is satisfied

$$\|\partial_x^\gamma \alpha_{\Theta x}(A)\Phi\| \leq C_\gamma (1 + |x|)^{m - \rho|\gamma|}, \qquad \forall \Phi \in \mathcal{D}^\infty(A).$$

Then, the deformed operator A_Θ is given as a well-defined oscillatory integral and is an essential self-adjoint operator on $\mathcal{D}^\infty(A)$.

* AM., J. Math. Phys. 56, (2015)

Deforming the Hamiltonian with Q_j*

Free Hamiltonian:

$$\boxed{H_0 = -P_j P^j/(2m) = -\Delta/(2m)}$$

Proposition

The scalar product $\langle \Psi, H_\theta \Phi \rangle$ is bounded,

$$|\langle \Psi, H_\theta \Phi \rangle| \leq C_B \, \|\Psi\|, \qquad \forall \Psi \in \mathscr{H}, \; \Phi \in \mathcal{E} \subseteq \mathscr{S}(\mathbb{R}^3),$$

Therefore the deformation for H_0 is well-defined and the result is

$$H_\theta \Phi = -\frac{1}{2m} \left(\vec{P} + i(\theta Q)^k [Q_k, \vec{P}] \right)^2 \Phi$$

* AM., J. Math. Phys. 55, 022302 (2014)

Example $Q = X$

Result for H_Θ

$$H_\Theta \Psi = -\frac{1}{2m}(P_j + \Theta_{jk} X^k)(P^j + \Theta^{jr} X_r)\Psi = -\frac{1}{2m} P_j^\Theta P_\Theta^j \Psi$$

Lemma

*Let the **deformation matrix** Θ_{ij} be given as,*

$$\Theta_{ij} = -(e/2)\,\varepsilon_{ijk} B^k,$$

*where B^k is a **magnetic field** (MF). Let the **deformation matrix** Θ_{ij} be*

$$\Theta_{ij} = m\,\varepsilon_{ijk}\,\Omega^k,$$

*where $\Omega^k = (2GM/r_{hs})\omega^k$ is a **gravitomagnetic field** (GMF).*

Other Effects generated by Deformation Quantization

Zeemaneffect (Magnetic and Gravitomagnetic)

Aharanov Bohm effect (Magnetic and Gravitomagnetic)

Lense Thirring effect

Curved Space-Times from Strict Deformations?

Sensible Requirements (Axioms) for a Theory of Quantum Gravity (QG)!

In QG the Metric should be operator-valued

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Obtain Moyal-Weyl Spacetime as a solution

Universal differential algebra (UDA)

The **UDA of forms** $\Omega(\mathscr{A}) = \bigoplus_p \Omega^p(\mathscr{A})$ is defined as:

$$\Omega^0(\mathscr{A}) = \mathscr{A}$$

and space of one-forms $\Omega^1(\mathscr{A})$ generated by $d : \mathscr{A} \to \Omega^1(\mathscr{A})$,

$$d^2 = 0, \qquad d(a_1 a_2) = (da_1)a_2 + a_1 da_2, \qquad \forall a_1,\, a_2 \in \mathscr{A}.$$

Moreover, the space $\Omega^p(\mathscr{A})$ is defined as

$$\Omega^p(\mathscr{A}) = \underbrace{\Omega^1(\mathscr{A}) \cdots \Omega^1(\mathscr{A})}_{p-times}.$$

Let us start with a commutative algebra $\mathscr{A}_c$ generated by

$$[\hat{x}^\mu, \hat{x}^\nu] = 0.$$

By applying the operator d on the commutator relations of $\mathscr{A}_c$

$$[d\hat{x}^\mu, \hat{x}^\nu] + [\hat{x}^\mu, d\hat{x}^\nu] = 0.$$

A general solution is given by

$$[\hat{x}^\mu, d\hat{x}^\nu] = \sum_{\sigma=0}^{n} C^{\mu\nu}_{\ \ \sigma}\, d\hat{x}^\sigma,$$

Lemma

The constants $C^{\mu\nu}_{\ \ \sigma}$ are symmetric in the first two indices and fulfill

$$\sum_{\sigma=0}^{n} \left(C^{\mu\nu}_{\ \ \sigma} C^{\lambda\sigma}_{\ \ \kappa} - C^{\lambda\nu}_{\ \ \sigma} C^{\mu\sigma}_{\ \ \kappa} \right) = 0$$

in order to be consistent.

Representation

Lemma

Let a consistent differential algebra be given as

$$[\hat{x}^\mu, d\hat{x}^\nu] = ia^\mu \delta^{\mu\nu} d\hat{x}^\nu.$$

Then, a faithful *-representation of $\mathscr{A}_c$, denoted by $\pi : \mathscr{A}_c \to \mathscr{L}^2(\mathbb{R}^d)$, is given on $\mathcal{D}(\pi) = \mathscr{S}(\mathbb{R}^d)$ by

$$\pi(\hat{x}^\mu) = \frac{a^\mu}{2}(Q^{\mu+1}P^{\mu+1} + P^{\mu+1}Q^{\mu+1}), \qquad a^\mu \neq 0.$$

A representation of the universal differential as derivation is defined by

$$\pi(d\hat{x}^\mu) := iq_b\,[P^b, \pi(\hat{x}^\mu)], \qquad q^\mu \neq 0.$$

It obeys Leibniz and satisfies $\pi(d^2\hat{x}^\mu) = 0$.

Deformation I

Definition

Let the deformed line-element, denoted by ds^2_Θ, be defined as

$$ds^2_\Theta := (\eta_{\mu\nu}\, d\hat{x}^\mu d\hat{x}^\nu)_\Theta,$$

where the deformation is performed by using the unitary operators,

$$U(p) = \exp(ip_\mu \pi(\hat{x}^\mu)), \qquad \forall p \in \mathbb{R}^d,$$

and the deformed differentials are defined as,

$$(d\hat{x}^\mu d\hat{x}^\nu)_\Theta := \pi^{-1}(\pi(d\hat{x}^\mu)\pi(d\hat{x}^\nu))_\Theta.$$

In order to ease readability we define the representations as follows,

$$\pi(\hat{x}^\mu) =: X^\mu, \qquad \pi(d\hat{x}^\mu) =: dX^\mu,$$

Outcome of Deformation I

The warped convoluted line-element:

$$(dX^{\mu})^2_{\Theta}\Phi = (2\pi)^{-d} \lim_{\epsilon \to 0} \iint dx\, dy\, e^{-ixy}\, \chi(\epsilon x, \epsilon y) U(y) \alpha_{\Theta x} (dX^{\mu})^2 \Phi$$

Theorem

By using the representation of differential algebra we obtain a well-defined deformed differential operator $(dX^{\mu})^2_{\Theta}$ on $\mathscr{S}(\mathbb{R}^d)$ given by

$$(dX^{\mu})^2_{\Theta} = e^{-2a_{\mu}(\Theta X)_{\mu}} (dX^{\mu})^2.$$

Conformal Flat Space-Times

Theorem

The deformed differential algebra gives the following warped convoluted line-element

$$(ds^2)_\Theta = \eta_{\mu\nu} d\hat{x}^\mu_\Theta d\hat{x}^\nu_\Theta = (\eta_{\mu\nu})_\Theta \, d\hat{x}^\mu d\hat{x}^\nu,$$

where from the deformation of the flat line-element we obtain the ***curved space-time metric*** $(\eta_{\mu\nu})_\Theta = e^{-2a_\mu (\Theta \hat{x})_\mu} \eta_{\mu\nu}$.

Friedmann-Robertson-Walker space-time

Theorem

Let the parameters of the model be given as,

$$2a_i\Theta_{0i} = H, \qquad \Theta_{ij} = 0,$$

where H is the Hubble parameter. Then, the deformed line-element gives a deformed ***Friedmann-Robertson-Walker space-time****, i.e.*

$$(ds^2)_\Theta = e^{-2a_0(\Theta\hat{x})_0} d\hat{t}^2 - e^{-H\hat{t}} d\hat{\mathbf{x}}^2$$

For case where $a_0\Theta_{0i} \approx 0$, the well-known undeformed FRW is obtained

$$(ds^2)_\Theta = d\hat{t}^2 - e^{-H\hat{t}} d\hat{\mathbf{x}}^2.$$

NC Space-time as Concistency Condition

Demand **centrality** of the metric tensor $\Longrightarrow$ Mathematical reasoning: Differential Geometry in NCG is defined only for central bi-modules!

Assume **Moyal-Weyl type** of non-commutativity between space and time,

$$[\hat{x}^0, \hat{x}^j] = i\,\Omega^{0j}, \qquad \Omega^{0j} \in \mathbb{R}^n$$

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Theorem

Let the deformation matrix be given by $\Theta_{0j} = \Theta e_j$. Then, for metrics $g_{\mu\nu} = (\eta_{\mu\nu})_\Theta$ obtained by deformation, ***centrality*** *is fulfilled by assuming the non-commutativity of the Moyal-Weyl plane*

$$[\hat{x}^0, \hat{x}^j] = i\,\Omega^{0j} = i\Omega e^j$$

with the additional condition $\Omega^{-1} = n\Theta$.

Thank you for your attention!